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Maximum Thrust Nozzles

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Introduction

THIS Note shows how the shape of an axisymmetrical nozzle can be found which optimizes the thrust for a given length l. We suppose the flow to take place in the meridian plane (see Fig. 1) with abscissa x and ordinate y. The method used requires an expansion curve, TB, be given as well as the input flow into the nozzle along a right going characteristic intersecting TB at C and the x axis at E. What must be found to obtain the desired shape is the flow along a left going characteristic of the nozzle flow intersecting CE at I and $x = l$ at F. This characteristic which varies along CIE on one end and along $x = l$ on the other is to afford a maximum to the thrust integral. After obtaining the flow along IF we have data along two characteristics so that a Goursat problem can be solved. Applying the total mass flow to this solution yields the nozzle shape. This work generalizes Rao's[1] one variable end point approach and contains his special solution.

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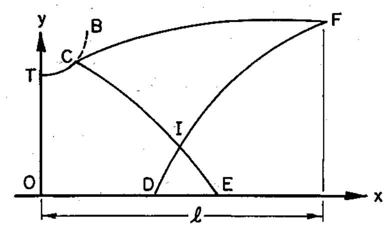

Fig. 1 Meridian plane of the nozzle.

Compressible Flow Results

Next we state results from irrotational steady supersonic flow of an ideal gas with no shock waves. The thrust, conservation of mass, nozzle length and compatibility condition are given respectively, as

$$\frac{T}{2\pi} = G(y_I) + \int_I^F y\left[p - p_0 + \frac{\rho w^2 \sin\alpha \cos\theta}{\sin(\theta + \alpha)}\right]dy \tag{1}$$

$$m(y_F) = \beta(y_I) + \int_I^F \frac{y\rho w \sin\alpha}{\sin(\theta + \alpha)}dy = 0 \tag{2}$$

$$l = x_c + \int_I^F \cot(\theta + \alpha)dy + L(y_I) \tag{3}$$

$$\dot{\theta} - [(\cot\alpha)/w]\dot{w} + \sin\alpha\sin\theta/[y\sin(\theta + \alpha)] = 0 \tag{4}$$

where $\cdot = d/dy$, w is the speed, θ is the inclination of the velocity, α is the Mach angle and p_0 is the ambient pressure. Let f_1, f_2, and f_3 be defined as the explicit integrands given in Eqs. (1), (2), and (3), respectively. Then $G(y_I)$ and $\beta(y_I)$ are integrands along EI and $L(y_I)$ is an integral along CI of f_1, f_2 and f_3, respectively, when α is replaced by $-\alpha$.

Variational Procedure

The variational method allows the use of Lagrange multipliers λ_1, λ_2, and $\lambda_4(y)$ to incorporate the constraints (2), (3) and (4). Thus we seek an extremal for the integral

$$\Phi = \int_I^F F(y,w,\dot{w},\dot{\theta},\theta,\lambda_1,\lambda_2,\lambda_4)dy + \gamma(y_I,\lambda_1,\lambda_2) \tag{5}$$

The necessary conditions resulting from Eq. (5) are the two Euler equations for variables w and θ and the constraint (4).

At the variable end I, we suppose that y_I is free but w and θ are fixed in the sense that they must be continuous at the intersection I. This yields the transversality condition at I as

$$\frac{y_I\rho w^2 \sin 2\alpha}{\sin(\theta + \alpha)\sin(\theta - \alpha)}\left\{\frac{\sin 2\theta}{2} + \frac{\lambda_1}{w}\sin\theta - \frac{\lambda_2}{y_I\rho w^2}\right\} + \lambda_4\left\{2\Theta - \frac{\sin^2\alpha\sin 2\theta}{y_I\sin(\theta + \alpha)\sin(\theta - \alpha)}\right\} = 0 \tag{6}$$

where $\Theta(y)$ is the velocity inclination along CE. The condition at F is

$$t(y_F) = f + \lambda_1 f_1 + \lambda_2 f_2 = 0 \quad \text{and} \quad \lambda_4(y_F) = 0 \tag{7}$$

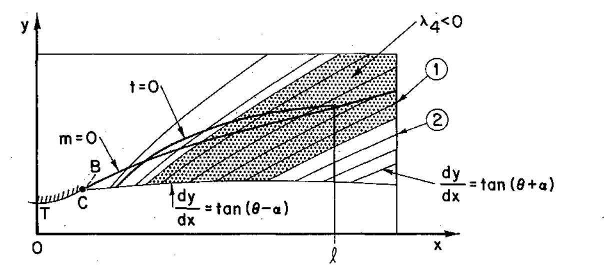

Fig. 2 Samples of characteristics with $\lambda_4 \neq 0$ which are solutions of Euler equations.

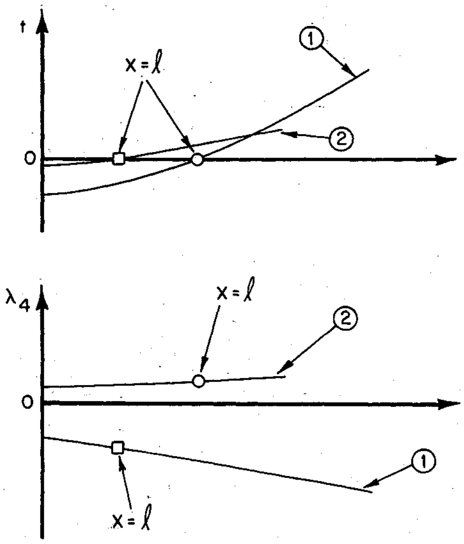

Fig. 3 Typical curves t and λ_4 plotted over the arc lengths of the characteristics No. 1 and 2 of Fig. 2.

Solutions for $\lambda_4 \not\equiv 0$

The Euler equations and constraint (4) constitute a system of first-order ordinary differential equations for the unknown functions w, θ, λ_4. The values of w and θ are fixed for any initial point I. If we solve the Euler equations for λ_4 and use Eq. (6) we obtain a linear relation between λ_1 and λ_2, i.e., we have for each initial point I a one-parameter family of solutions. The corresponding characteristics, $dy/dx = \tan(\theta + \alpha)$, all pass through the point I. The free parameter of these solutions can be chosen so that one of the boundary conditions at F is satisfied.

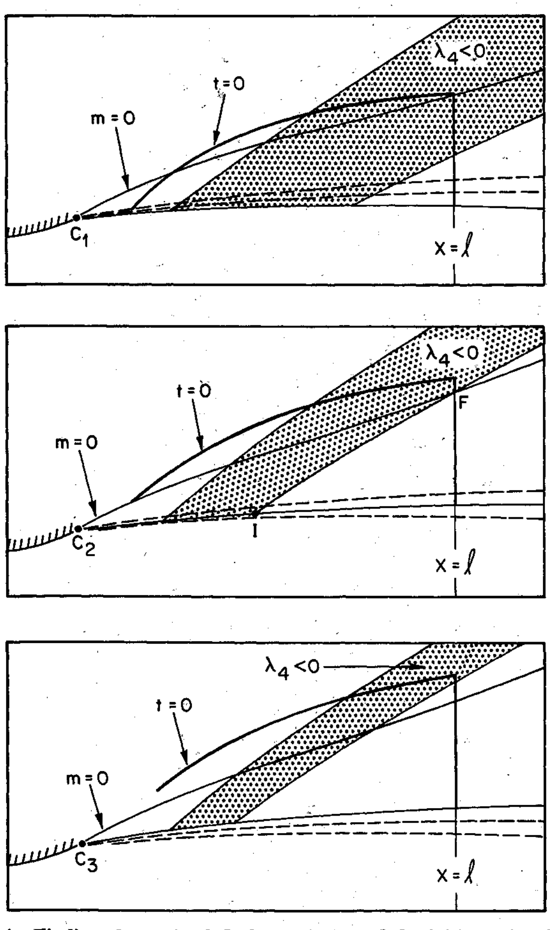

Fig. 4 Finding the optimal IF by variation of the initial point C and using the same set of solutions as in Fig. 2.

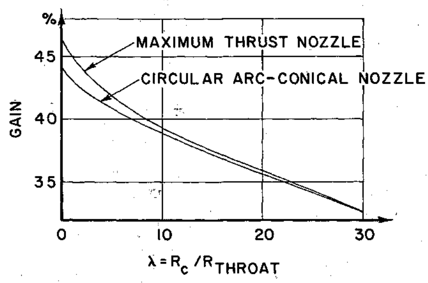

Fig. 5 Comparison of performances of maximum thrust nozzles with circular arc–conical nozzles with equal exit and throat areas.

Figure 2 shows results of calculations where we assumed a specific heat ratio of 1.24, a stagnation density of 8.53 kg/m^3, a stagnation pressure of 7.16 N/m^2, zero ambient pressure, a length to throat radius ratio of 10 and a throat curvature radius to throat radius ratio λ of 3. The free parameter of the solutions was determined such that $|t|$ was minimized at $x = l$. Also for each solution, the point was computed at which Eq. (2) was satisfied. The right hand intersection of the curves $t = 0$ and $m = 0$ in Fig. 2 is the end point F of the optimal IF, if at that intersection the second condition in Eq. (7) is satisfied. In the case shown in Fig. 2, $\lambda_4 < 0$ at the intersection. Figure 3 shows typical curves t and λ_4, corresponding to two characteristics of Fig. 2.

If the initial point C is varied along the arc TB one might find such a position of C that the intersection of $m = 0$ and $t = 0$ at $x = l$ coincides with $\lambda_4 = 0$. This remark is illustrated by Fig. 4. The position of C_2 which is between C_1 and C_3 yields the optimal IF.

Solutions with $\lambda_4 \equiv 0$

The one parameter family of solutions corresponding to each initial point I contains a special solution with $\lambda_4 \equiv 0$. Under restrictions the Euler equations yield explicit solutions for λ_1 and λ_2 which satisfy Eq. (4), and hence constitute a special solution to the necessary conditions for Φ. Moreover condition Eq. (6) is satisfied, while the first condition in Eq. (7) yields an expression for the pressure and the second is satisfied identically. These expressions for λ_1, λ_2 and the pressure correspond to Rao's[1] Eqs. (25) and (26). Hence we have shown that Rao's solution is a special member of the family of solutions corresponding to initial point I.

Comparison with Conical Nozzles

The performance of the maximum thrust nozzles and the performance of comparable conical nozzles are shown in Fig. 5. The "comparable" conical nozzles have the same value λ and the same exit and throat areas as the corresponding maximum thrust nozzles. Hence both nozzle types generally would fit into the same motors as far as their size is concerned. In order to compare the efficiency of the nozzles we define as the "gain of the nozzle" the ratio $(T_E - T_T)/T_T$ where T_T is the thrust at the throat and T_E is the thrust at the exit. The two gain curves in Fig. 5 show that the gain improvement by a contoured nozzle is of the order of a few per cent only. Similar computations for shorter nozzles, i.e., smaller length to throat radius ratio, show even smaller gain improvement. Hence the more complicated and expensive manufacturing of contoured nozzles will generally by justified only if the utilization of the propellant is critical, e.g., in space rockets. In most other cases the use of conical nozzles will be appropriate from the view point of production costs and nozzle efficiency. However, the half-angle of the conical section and the radius of curvature must be chosen in a specific way to obtain a nozzle with a thrust close to the optimum.

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